AD 294 987

UNCLASSIFIED

63-2-3

UNIVERSITY OF

MARYLAND

THE INSTITUTE FOR FLUID DYNAMICS

and

APPLIED MATHEMATICS

[illegible] supported by the
Applied Mathematics Division, AFOSR,
SRMA
under Contract/Grant AF-AFOSR-62-454

Technical Note BN-306 December, 1962

OPERATORS WHICH GENERATE HARMONIC FUNCTIONS IN THREE-VARIABLES*

by

R. P. Gilbert

University of Maryland
College Park, Maryland

* This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under grant-AFOSR 62-454.

ABSTRACT

In this paper a new construction of the Bergman-Whittaker operator is presented. Another operator, which transforms functions of two complex variables into harmonic functions in three variables is introduced along with its inverse operator. A theorem connecting the singularities of analytic functions with those of harmonic functions is given with an illustration. These methods are extended to a class of singular hyperbolic equations.

I. Introduction

Harmonic functions in three-variables may be generated by means of the Bergman-Whittaker integral operator, $B_3[f]$, which maps analytic functions of three variables $H(X)$.[1][2][3]

$$H(\mathbf{X}) = B_3[f], \quad B_3[f] \equiv \frac{1}{2\pi i}\int_{\mathcal{L}} f(t,\zeta)\frac{d\zeta}{\zeta},$$

$$t = \left[-(x_1 - i x_2)\frac{\zeta}{2} + x_3 + (x_1 + i x_2)\frac{1}{2\zeta}\right], \tag{1}$$

$$\|\mathbf{X} - \mathbf{X}^0\| < \epsilon, \quad \mathbf{X} \equiv (x_1, x_2, x_3), \quad \mathbf{X}^0 \equiv (x_1^0, x_2^0, x_3^0)$$

where $\mathcal{L}$ is a closed differentiable arc in the ζ - plane, and $\epsilon > 0$ is sufficiently small.

Certain classes of harmonic functions in three-variables may be generated, however, by an operator which maps functions of a single complex variable. For instance the Bergman-Sommerfiled operator[4],

$$H(\mathbf{X}) = S_3[f], \quad S_3[f] \equiv \int_{\mathfrak{d}} \frac{f(\zeta)\, d\zeta}{\left([x_1 - u_1(\zeta)]^2 + [x_2 - u_2(\zeta)]^2 + [x_3 - u_3(\zeta)]^2\right)^{1/2}} \tag{2}$$

where $\mathfrak{d}$ is either an open or closed curve in the ζ - plane, and the $u_K(\zeta)$ are real functions of ζ, etc..

It is of interest, in the study of analytic properties of harmonic functions in three variables to have different operators available. The reason for this is that by using different operators we are frequently able to transplant some other properties of analytic functions for certain classes of harmonic functions. In this paper we shall introduce several new operators for harmonic functions in three variables and study some of their properties.

II. A New Derivation of the Whittaker-Bergman Operator

In previous works [1] [4] [5] the Whittaker-Bergman operator $B_3[f]$ has been obtained by various methods. We shall present here a somewhat different approach by first introducing the kernel,

$$\hat{K}\left(\frac{r}{s}, \cos\theta, \frac{e^{i\varphi}}{i\zeta}\right) \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \frac{n!}{(n+m)!} \left(\frac{r}{s}\right)^n P_n^m(\cos\theta) \frac{e^{im\varphi}}{(i\zeta)^m}, * \tag{2.1}$$

which converges for $\left|\frac{r}{s}\right| < 1$, $1-\delta < |\zeta| < 1+\delta$, $\delta > 0$. Now, if we define the analytic function of two complex variables, $f(s, \zeta)$, and the harmonic function of three variables, $V(r, \theta, \varphi)$, as follows,

$$f(s,\zeta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} a_{nm}\, s^n \zeta^m, \quad |s| < \epsilon, \ 1-\delta < |\zeta| < 1+\delta, \tag{2.2}$$

and

$$V(r,\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} a_{nm} \frac{n!}{(n+m)!} r^n P_n^m(\cos\theta) \frac{e^{im\varphi}}{i^m}, \quad r < \epsilon. \tag{2.3}$$

---

* The $P_n^m(\xi)$ are associated legendre functions.

Then, one has formally, that

$$V(r,\theta,\varphi) = -\frac{1}{4\pi^2}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\int_{|s|=2r}\frac{ds}{s}\,\hat{K}\left(\frac{r}{s},\cos\theta,\frac{e^{i\varphi}}{i}\right)f(s,\zeta)\,. \tag{2.4}$$

Recalling the generating function for the spherical harmonics [5][7],

$$t^n \equiv \left\{-(x_1 - i x_2)\frac{\zeta}{2} + x_3 + (x_1 + i x_2)\frac{1}{2\zeta}\right\}^n$$

$$= \sum_{m=-n}^{+n} \frac{n!}{(n+m)!}\, r^n\, P_n^m(\cos\theta)\,\frac{e^{im\varphi}}{(i\zeta)^m}\;, \tag{2.5}$$

(where $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$, $x_3 = r\cos\theta$),
we see that, providing $\left|\frac{t}{r}\right| < 1$, one has

$$\hat{K}\left(\frac{r}{s},\cos\theta,\frac{e^{i\varphi}}{i\zeta}\right) = \sum_{n=0}^{\infty}\left(\frac{t}{s}\right)^n = \frac{s}{s-t}\,. \tag{2.6}$$

Consequently, if we choose $|t| < \frac{\epsilon}{2}$, then by the Cauchy formula we have

$$V(r,\theta,\varphi) = -\frac{1}{4\pi^2}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\int_{|s-t|=\frac{\epsilon}{2}}\frac{f(s,\zeta)}{s-t}\,ds = \frac{1}{2\pi i}\int_{|\zeta|=1} f(t,\zeta)\,\frac{d\zeta}{\zeta}\,. \tag{2.7}$$

This is the Whittaker-Bergman operator defining a harmonic function in the small of the origin.

III. An Operator which Generates Harmonic Functions Having Certain Symmetry Properties

As in the last section we shall introduce a kernel by which one may generage harmonic functions as integral transforms of analytic functions in two variables. For instance, let

$$K_1\left(\frac{r}{s}, \cos\theta, \frac{e^{i\varphi}}{\zeta}\right) \equiv \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{2^m m!}{(2m)!} \left(\frac{r}{s}\right)^n P_n^m(\cos\theta)\left(\frac{e^{i\varphi}}{\zeta}\right)^m, \tag{3.1}$$

then if $f(s,\zeta)$ is defined in the bi-cylinder $\{|s| \leq 1, |\zeta| \leq 1\}$ as

$$f(s,\zeta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{nm} s^n \zeta^m , \tag{3.2}$$

one has by the Cauchy formula for several variables, that

$$U(r, \cos\theta, e^{i\varphi}) \equiv \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{nm} \frac{2^m m!}{(2m)!} r^n P_n^m(\cos\theta) e^{im\varphi}, \tag{3.3}$$

has the integral representation,

$$U(r, \cos\theta, e^{i\varphi}) = \frac{-1}{4\pi^2} \int_{|s|=1} \frac{ds}{s} \int_{|\zeta|=1} K_1\left(\frac{r}{s}, \cos\theta, \frac{e^{i\varphi}}{\zeta}\right) f(s,\zeta) \frac{d\zeta}{\zeta}. \tag{3.4}$$

In order to sum $K_1\left(\frac{r}{s}, \cos\theta, \frac{e^{i\varphi}}{\xi}\right)$, we consider the generating function for the Legendre polynomials[ ]

$$\frac{1}{(1-2r\xi+r^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(\xi)\, r^n \quad , \quad |r|<1 , \tag{3.5}$$

and differentiate m-times with respect to ξ ; we obtain[ ]

$$\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2m-1)\, r^m}{(1-2r\xi+r^2)^{m+1/2}} = \sum_{n=m}^{\infty} r^n \frac{d^m}{d\xi^m} P_n(\xi) = \sum_{n=m}^{\infty} (-1)^m (1-\xi^2)^{-m/2} P_n^m(\xi)\, r^n ,$$

(3.6)

since $$P_n^m(\xi) = (-1)^m (1-\xi^2)^{m/2} \frac{d^m}{d\xi^m} P_n(\xi).$$

Now, if we multiply both sides by $\left(\frac{\eta}{\xi}\right)^m$, and sum from o to ∞ over m, (replacing r by r/s), one has*

$$\frac{1}{\sqrt{1-2\frac{r}{s}\xi+\frac{r^2}{s^2}}} \sum_{m=0}^{\infty} \left(\frac{-\eta r s\sqrt{1-\xi^2}}{\xi(s^2-2rs\xi+r^2)}\right)^m$$

$$= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \frac{2^m\, m!}{(2m)!} \left(\frac{r}{s}\right)^n P_n^m(\xi)\left(\frac{\eta}{\xi}\right)^m \tag{3.7}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{2^m\, m!}{(2m)!} \left(\frac{r}{s}\right)^n P_n^m(\xi)\left(\frac{\eta}{\xi}\right)^m .$$

---

* We may rearrange orders of summation since the series is absolutely convergent.

Consequently,

$$K_1\left(\frac{r}{s}, \xi, \frac{\eta}{s}\right) = \frac{s}{\sqrt{s^2-2rs\xi+r^2}}\left[\frac{}{1+\dfrac{\eta r s\sqrt{1-\xi^2}}{\zeta(s^2-2rs\xi+r^2)}}\right]$$

$$= \frac{s\sqrt{s^2-2rs\xi+r^2}}{\zeta(s^2-2rs\xi+r^2)+\eta r s\sqrt{1-\xi^2}}, \qquad (3.8)$$

providing $\left|\dfrac{\eta r s\sqrt{1-\xi^2}}{\zeta(s^2-2rs\xi+r^2)}\right| \leq 1$, which for $|\eta| = |\zeta| = |s| = 1$,

is satisfied for $|r| < 1/3$. We may then continue analytically for values of r , such that $|r| > 1/3$. Consequently, the representation (3.4) may be expressed as

$$U(r,\xi,\eta)$$

$$= \frac{-1}{4\pi^2}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\int_{|s|=1}\frac{ds\, F(s,\zeta)\sqrt{s^2-2rs\xi+r^2}}{\zeta(s^2-2rs\xi+r^2)+\eta r s\sqrt{1-\xi^2}}. \qquad (3.9)$$

in a sufficiently small neighborhood of the origin; in Cartesian coordinates

(3.9) has the form, $\left(\xi = \dfrac{x_3}{r}\, , \ \eta = \sqrt{\dfrac{x_1+ix_2}{x_1-ix_2}}\, , \ r = +\sqrt{x_1^2+x_2^2+x_3^2}\right)$,

$$H(x_1, x_2, x_3)$$

$$= -\frac{1}{4\pi^2}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\int_{|s|=1}\frac{ds\;F(s,\zeta)\sqrt{x_1^2+x_2^2+(x_3-s)^2}}{\left(x_1+\frac{s}{2\zeta}\right)^2+\left(x_2+\frac{is}{2\zeta}\right)^2+(x_3+s)^2}\,. \tag{3.10}$$

It is possible to find an integral representation for (3.3), however, in terms of just a single Cauchy integral. We return to (3.6), multiply by $(\eta/\zeta)^m$, replace r by r/s, and then perform a contour integration over $|s|=1$;

$$\frac{2^m m!}{(2m)!}\, r^K P_K^m(\xi)\eta^m = \frac{1}{2\pi i}\int_{|s|=1}\left[\frac{-\eta r\sqrt{1-\xi^2}}{s\left(1-2\frac{r}{s}\xi+\frac{r^2}{s^2}\right)}\right]^m\frac{s^K\,ds}{\sqrt{s^2-2rs\xi+r^2}}, \tag{3.11}$$

$K \geq m$.

This suggests that we consider the function of two complex-variables

$$F(s,\sigma) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{nm}\, s^n\sigma^m$$

, where

$$\sigma = \frac{-\eta r\sqrt{1-\xi^2}}{s\left(1-2\frac{r}{s}\xi+\frac{r^2}{s^2}\right)} = \frac{-s(x_1+ix_2)}{x_1^2+x_2^2+(x_3-s)^2}\,. \tag{3.12}$$

One then has formally, that

$$H(x_1, x_2, x_3) = \frac{1}{2\pi i} \int_{|s|=1} F(s,\sigma) \frac{ds}{\sqrt{x_1^2 + x_2^2 + (x_3 - s)^2}}, \tag{3.13}$$

where σ is defined by (3.12). We shall refer to the integral representation (3.13) as the operator, $P_3^*[F]$, and to $F(s,\sigma)$ as the P_3^* - associate of $H(X)$, $X \equiv (x_1, x_2, x_3)$.

One interesting application of the integral operator method is that one may transplant certain properties of analytic functions into properties concerning solutions of partial differential equations. This has been done in the case of the operator, $B_3[F]$, by Bergman [1][2][3][4], Kreyszig [10][11], Gilbert [5][12], and White [13][14]. We state here a theorem concerning the singularities of the harmonic function - element, which is defined in the small of the origin by (3.13); the proof is similar in structure to an earlier one by the author [5][12][15].

<u>Theorem I:</u>

<u>Let</u> $\mathcal{Z}^3 \equiv E\{\phi(s,\sigma) \equiv \psi(X;s) = 0\}$ <u>be the singularity manifold of</u> $F(s,\sigma)$, <u>then the harmonic function-element defined by</u> $H(X) = P_3^*[F]$ <u>is regular at all points</u> X, <u>such that</u>

$$X \notin \left[E\{\psi = 0\} \cap E\left\{\frac{\partial \psi}{\partial s} = 0\right\}\right] \cup E\{x_1^2 + x_2^2 = 0\}. \tag{3.14}$$

If $\chi \equiv x_1^2 + x_2^2 + (x_3 - s)^2 \neq 0$, then the only singularities of the integrand occur for $\Psi(\mathbf{X}; s) = 0$. From an earlier result of the author [5][15] it follows that singularites of $H(\mathbf{X})$ may occur for $\mathbf{X} \in \left[E\{\Psi = 0\} \cap E\left\{\frac{\partial \Psi}{\partial s} = 0\right\}\right]$. If $\chi = 0$ (which means $\sigma = \infty$) singularities <u>may</u> also occur for

$$\mathbf{X} \in \left[E\{\chi = 0\} \cap E\left\{\frac{\partial \chi}{\partial s} = 0\right\}\right] \equiv E\{x_1^2 + x_2^2 = 0\} .$$

<u>Illustration:</u> Let $F(s, \sigma) \equiv \dfrac{1}{s - \sigma}$, then if $\chi \neq 0$, we may express the singularity manifold of $F(s, \sigma)$, as,

$$\Psi(\mathbf{X}; s) \equiv s \left([x_1 + \tfrac{1}{2}]^2 + [x_2 + \tfrac{i}{2}]^2 + [x_3 - s]^2 \right) = 0 . \qquad (3.15)$$

Eliminating s between $\Psi(\mathbf{X}; s) = 0$ and $\dfrac{\partial \Psi}{\partial s} = 0$, yields

$$\Lambda^2 \equiv E\left\{(x_1 - \tfrac{1}{2})^2 + (x_2 + \tfrac{i}{2})^2 + x_3^2 = 0\right\} \cup E\left\{(x_1 + \tfrac{1}{2})^2 + (x_2 + \tfrac{i}{2})^2 = 0\right\}. \qquad (3.16)$$

The possible singularities of $H(\mathbf{X})$, then must lie on

$M^2 \equiv \Lambda^2 \dot{\cup} E\{x_1^2 + x_2^2\}$. The restriction of M^2 to R^3 (real three-space) is $E\ \{x_1 = -\tfrac{1}{2}, 0 ; x_2 = 0 ; x_3 = \text{arbitrary real}\}$.

IV. An Inverse Operator for $\mathcal{P}_3^*[F]$.

In order to construct an integral transform, which maps the harmonic function,

$$H(\mathbf{X}) \equiv U(r,\xi,\eta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{nm}\,\frac{2^m m!}{(2m)!}\,r^n P_n^m(\xi)\,\eta^m ,$$

back onto its $\mathcal{P}_3^*$ - associate,

$$F(s,\sigma) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{nm}\, s^n \sigma^m ,$$

we first introduce the kernel,

$$K_2\left(\frac{u}{r},\xi,\frac{s}{\eta}\right) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(2n+1)(n-m)!}{(n+m)!}\,\frac{(2m)!}{2^m m!}\left(\frac{u}{r}\right)^n P_n^m(\xi)\left(\frac{s}{\eta}\right)^m . \qquad (4.1)$$

From the orthogonality property of the associated legendre functions,[8]

$$\int_{-1}^{+1} P_n^m(\xi)\,P_l^m(\xi)\,d\xi = \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!} , \qquad (4.2)$$

one has formally, that

$$F(u,\zeta) = \frac{1}{4\pi i} \int_{-1}^{+1} \left[\int_{|\eta|=1} \frac{d\eta}{\eta} K_2\left(\frac{u}{r}, \xi, \frac{\zeta}{\eta}\right) U(r, \xi, \eta) \right] d\xi . \tag{4.3}$$

We now investigate the series expression for $K_2(\frac{u}{r}, \xi, \frac{\zeta}{\eta})$, and proceed to do this in a purely formal manner. We first split $K_2(\frac{u}{r}, \xi, \frac{\zeta}{\eta})$ into two parts,

$$K_2\left(\frac{u}{r}, \xi, \frac{\zeta}{\eta}\right) = \sum_{n=0}^{\infty} (2n+1)\left(\frac{u}{r}\right)^n P_n(\xi)$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(2n+1)\ 2m}{2^m\ m!} B(n-m+1, 2m)\left(\frac{u}{r}\right)^n P_n^m(\xi)\left(\frac{\zeta}{\eta}\right)^m \tag{4.4}$$

$$= \frac{r^2(r^2-u^2)}{(r^2-2r\xi+u^2)^{3/2}} + 2\int_0^1 \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(2n+1)}{(m-1)!}\left(\frac{ut}{r}\right)^n P_n^m(\xi) \left[\frac{\zeta(1-t)^2}{2\eta t}\right]^m \frac{dt}{1-t},$$

where $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \displaystyle\int_0^1 t^{p-1}(1-t)^{q-1}\, dt$, when $Re\, p > 0$, $Re\, q > 0$.

We consider the sum,

$$S(\tau,\xi,\rho) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(2n+1)}{m!} \tau^n P_n^m(\xi) \rho^m , \quad \text{where}$$

$$\tau = \frac{ut}{r}, \quad \text{and} \quad \rho = \frac{\zeta(1-t)^2}{2\eta t}, \tag{4.5}$$

since one has

$$\rho \frac{\partial}{\partial \rho} \; S(\tau, \xi, \rho) \equiv \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(2n+1)}{(m-1)!} \, \tau^n P_n^m(\xi) \, \rho^m . \tag{4.6}$$

In order to formally sum (4.5) we consider the formula

$$\sum_{n=m}^{\infty} \frac{(2n+1)}{m!} \tau^n P_n^m(\xi) \rho^m \tag{4.7}$$

$$= \frac{(2m+1)(-1)^m (1-\xi^2)^{m/2} \tau^m}{(1-2\tau\xi+\tau^2)^{m+3/2}} \; \frac{\rho^m (2m)!}{2^m (m!)^2} (1-\tau^2) ,$$

which may be obtained by differentiating,

$$\frac{1-\tau^2}{(1-2\tau\xi+\tau^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) \tau^n P_n(\xi) \; , \; |\tau| < \delta, \tag{4.8}$$

$$\delta = \min \{ | \xi \pm (\xi^2 - 1)^{1/2} | \} .$$

with respect to ξ, m-times. By summing (4.7) with respect to m from o to ∞, and interchanging orders of summation we have

$$S(\tau, \xi, \rho) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(2n+1)}{m!} \tau^n P_n^m(\xi) \rho^m$$

$$= \frac{(1-\tau^2)}{(1-2\tau\xi+\tau^2)^{3/2}} \sum_{m=0}^{\infty} \frac{(2m+1)!}{2^m (m!)^2} \left(\frac{-\tau\rho\sqrt{1-\xi^2}}{1-2\tau\xi+\tau^2} \right)^m . \tag{4.9}$$

By means of the Legendre duplication formula [8] we may reduce the coefficients on the right-hand side of (4.9) as follows,

$$\frac{(2m+1)!}{(m!)^2\; 2^m} = \frac{\Gamma(2m+2)}{\Gamma(m+1)^2\; 2^m} = \frac{1}{m!}\left[\frac{2^{m+3}}{\sqrt{\pi}}\Gamma(m+3/2)\right]$$

$$= \frac{2^3\, 2^m}{\sqrt{\pi}}\;\frac{\Gamma(m+3/2)}{m!}\;. \qquad (4.10)$$

Consequently, whenever $\left|\frac{\tau\rho\sqrt{1-\xi^2}}{1-2\tau\xi+\tau^2}\right| < 1, \; (-1\leq\xi\leq 1)$, we have

$$S(\tau,\xi,\rho) = \frac{4\,(1-\tau^2)}{(1-2\tau\xi+\tau^2)^{3/2}}\cdot\frac{1}{\left(1+\frac{2\tau\rho\sqrt{1-\xi^2}}{1-2\tau\xi+\tau^2}\right)^{3/2}}$$

$$= \frac{4\,(1-\tau^2)}{\left(1-2\tau\left[\xi-\rho\sqrt{1-\xi^2}\right]+\tau^2\right)^{3/2}}\;. \qquad (4.11)$$

Consequently,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(2n+1)}{(m-1)!}\,\tau^n\,P_n^m(\xi)\,\rho^m = \frac{12\,\tau\rho\,(1-\tau^2)\sqrt{1-\xi^2}}{\left(1-2\tau\left[\xi-\rho\sqrt{1-\xi^2}\right]+\tau^2\right)^{5/2}}\,, \qquad (4.12)$$

and

$$K_2\left(\frac{u}{r},\xi,\frac{\zeta}{\eta}\right) = \frac{r(r^2-u^2)}{(r^2-2ur\xi+u^2)^{3/2}} + 12u^2r^2\xi\sqrt{1-\xi^2}\int_0^1 \frac{(1-t)(r^2-u^2t^2)\,dt}{\left[at^2+bt+c\right]^{5/2}},$$

(4.13)

where $a = u + \frac{r}{\eta}\sqrt{1-\xi^2}$, $b = -2ur\left(\xi + \frac{1}{\eta}\sqrt{1-\xi^2}\right)$, $c = r^2 + \frac{ur}{\eta}\sqrt{1-\xi^2}$.

If we rewrite $K_2\left(\frac{u}{r},\xi,\frac{\zeta}{\eta}\right)$ as $\int_0^1 J\left(\frac{u}{r},\xi,\frac{\zeta}{\eta};t\right)dt$,

$$\int_0^1 \left\{\frac{2t\,r(r^2-u^2)}{(r^2-2ur\xi+u^2)^{3/2}} + \frac{12u^2r^2\xi\eta^{-1}\sqrt{1-\xi^2}\,(1-t)(r^2-u^2t^2)}{\left[at^2+bt+c\right]^{5/2}}\right\}dt,$$

then the inverse operator has the representation,

$$F(u,\zeta) = P_3^{*-1}[U]$$

$$\equiv \frac{1}{4\pi i}\int_{-1}^{+1} d\xi \int_{|\eta|=1} \frac{d\eta}{\eta} \int_0^1 dt \left\{J\left(\frac{u}{r},\xi,\frac{\zeta}{\eta};t\right)U(r,\xi,\eta)\right\}. \qquad (4.14)$$

However, it is possible to evaluate $K_2\left(\frac{u}{r},\xi,\frac{\zeta}{\eta}\right)$ by means of the elementary integrals, which we list below [16]:

$$\int \frac{t^m}{T^{5/2}}\,dt = A(m)\,\frac{t^{m-1}}{T^{3/2}} + B(m)\int \frac{t^{m-1}}{T^{5/2}}\,dt + D(m)\int \frac{t^{m-2}}{T^{5/2}}\,dt,$$

$$\int \frac{\alpha t+\beta}{T^{5/2}}\,dt = \frac{-\alpha}{3a}\,\frac{1}{T^{3/2}} + \frac{a\beta - b\alpha}{a}\int \frac{dt}{T^{3/2}},$$

$$\int \frac{dt}{T^{5/2}} = (at+b)\left[\frac{2a/3}{(ac-b^2)^2}\,\frac{1}{T^{1/2}} + \frac{1}{3}\,\frac{1}{(ac-b^2)}\,\frac{1}{T^{3/2}}\right], \tag{4.15}$$

where $T = at^2+bt+c$, $A(m) = \frac{1}{(m-4)a}$, $B(m) = -\frac{(2m-5)b}{(m-4)a}$, and $D(m) = \frac{-(m-1)c}{(m-4)a}$.

Using (4.15) we obtain $I\left(\frac{u}{r}, \xi, \frac{\rho}{r}\right) \equiv \int_0^1 \frac{(1-t)(r^2-u^2t^2)\,dt}{T^{5/2}}$

$$= \left[\frac{1}{a^3T^{3/2}}\left(-a^2t^2 + (a^2-ab)\frac{t}{2} - \frac{1}{6}(4ac-b^2-ab) + \frac{r^2a^2}{3u^2}\right)\right]_0^1$$
$$+ \frac{1}{2a^3}\left(2a^2\frac{r^2}{u^2}(a+b) - b(4ac-b^2) - a^2b^2\right)\int_0^1 \frac{dt}{T^{5/2}}$$

$$= \left[\frac{1}{a^3T^{3/2}}\left(-a^2t^2 + \frac{a^2-ab}{2}t - \frac{1}{6}(4ac-b^2-ab) + \frac{r^2a^2}{3u^2}\right)\right.$$
$$\left. + \frac{1}{6}\,\frac{(at+b)}{(ac-b^2)}\left(2a^2\frac{r^2}{u^2}(a+b) - b(4ac-b^2) - ab^2\right)\right]_0^1 \tag{4.16}$$
$$+ \left[\frac{1}{3a^2T^{1/2}}\left(2a^2\frac{r^2}{u^2}(a+b) - b(4ac-b^2) - ab^2\right)\frac{(at+b)}{(ac-b^2)^2}\right]_0^1,$$

and consequently $K_2\left(\frac{u}{r}, \xi, \frac{\zeta}{\eta}\right)$ may be expressed in terms of (4.16) as

$$K_2\left(\frac{u}{r}, \xi, \frac{\zeta}{\eta}\right) = \frac{r(r^2-u^2)}{(r^2-2ur\xi+u^2)^{3/2}} + \frac{12u^2r^2}{\eta}\xi\sqrt{1-\xi^2}\, I\left(\frac{u}{r}, \xi, \frac{\zeta}{\eta}\right). \quad (4.17)$$

V. A Partial Differential Equation in Three-Variables which Is Related to the G.A.S.P.T. Equation

We may make a natural extension of the operators introduced in the previous sections if we consider the generating function for the ultra-spherical harmonics [8] (Gengenbauer polynomials),

$$\frac{1}{(1-2r\xi+r^2)^{\lambda}} = \sum_{n=0}^{\infty} r^n C_n^{\lambda}(\xi) \quad , \quad |r| < \delta , \quad (5.1)$$

$$\delta = \min\left\{ \left| \xi \pm (\xi^2-1)^{1/2} \right| \right\}.$$

We differentiate this expression with respect to ξ, m-times, and make use of the identity [8],

$$\frac{d^m}{d\xi^m} C_n^{\lambda}(\xi) = (2\lambda)^m C_{n-m}^{\lambda+m}(\xi) \; ; \quad (5.2)$$

one obtains

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} \frac{(-1)^m r^m/\lambda^m}{(1-2r\xi+r^2)^{\lambda+m}} = \sum_{n=m}^{\infty} C_{n-m}^{\lambda+m}(\xi)\, r^n . \quad (5.3)$$

Summing over m from 0 to ∞, and inverting the order of summation on the right-hand side we have,

$$\sum_{m=0} \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)\, m!}\left(\frac{-rt}{\lambda}\right)^m \frac{1}{(1-2r\xi+r^2)^{\lambda+m}} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{r^n}{m!} C_{n-m}^{\lambda+m}(\xi)\, t^m$$

$$= \frac{1}{(1-2r\xi+r^2)^{\lambda}} \sum_{m=0}^{\infty} \frac{\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)}\left[\frac{-rt}{\lambda(1-2r\xi+r^2)}\right]^m \tag{5.4}$$

$$= \frac{1}{(1-2r\xi+r^2)^{\lambda}} \cdot \frac{1}{\left[1+\frac{rt}{\lambda(1-2r\xi+r^2)}\right]^{\lambda}} = \frac{1}{\left(1-2r\left[\xi-\frac{t}{2\lambda}\right]+r^2\right)^{\lambda}}.$$

We shall now show that the functions

$$u_{nm}(x,y,t) \equiv r^n C_{n-m}^{\lambda+m}(\xi)\, t^m, \quad (n=0,1,\ldots;\ m=0,1,2,\ldots,n)$$

$$\xi = \frac{x}{r}, \quad r = +\sqrt{x^2+y^2}, \tag{5.5}$$

satisfy the partial differential equation,

$$L_{2\lambda}[u] = \frac{x^2 t}{x^2+y^2}\frac{\partial}{\partial t}\left(t\,u_t + \frac{2}{x}u_x - \frac{2}{y}u_y + \frac{2\lambda}{x^2}u\right), \tag{5.6}$$

where $L_{2\lambda}[u] \equiv \Delta u + \frac{2\lambda}{y} u_y$ is the differential operator of Generalized Axially Symmetric Potential Theory (GASPT) [17][18]*.

First we notice, that $W_{nm} = r^{n-m} C^{\lambda+m}_{n-m}(\xi)$ satisfies

$$L_{2(\lambda+m)}[W] \equiv \Delta W + \frac{2(\lambda+m)}{y} W_y \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) + \frac{1}{r^2}\left[(1-\xi^2)\frac{\partial^2 W}{\partial \xi^2} - \xi\frac{\partial W}{\partial \xi}\right] + 2(\lambda+m)\left[r\frac{\partial W}{\partial r} - \xi\frac{\partial W}{\partial \xi}\right] = 0. \qquad (5.7)$$

Then, since $W_{nm} = (tr)^{-m} u_{nm}$, we realize that the must satisfy

$$r^2 L_{2\lambda}[u_{nm}] = 2m\xi\frac{\partial u}{\partial \xi} + m(2\lambda+m)u, \qquad (5.8)$$

and the class of function $\{r^n C^{\lambda+m}_{n-m}(\xi)\, t^m\}$ $(0 \le m \le n;\ n = 0,1,2,\dots)$ satisfy the partial differential equation

$$r^2 L_{2\lambda}[u] = 2t\xi\frac{\partial^2 u}{\partial t\,\partial \xi} + t\frac{\partial}{\partial t}\left(t\frac{\partial u}{\partial t}\right) + 2\lambda t\frac{\partial u}{\partial t}. \qquad (5.9)$$

When we transform to Cartesian coordinates, this becomes (5.6).

---

*The concept of fractional-dimensional potential theory was first introduced by A. Weinstein.

We realize now, that we may introduce an integral operator $\mathcal{G}_3[F]$, which will transform functions of two complex variables,

$$F(u, \zeta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{nm} u^n \zeta^m ,$$

onto solutions of the partial differential equation (5.6),

$$u(x, y, t) \equiv V(r, \xi, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{nm} r^n C_{n-m}^{\lambda+m}(\xi) \frac{t^m}{m!} ,$$

by introducing as the generating kernel,

$$K_3\left(\frac{r}{u}, \xi, \frac{t}{\zeta}\right) \equiv \frac{1}{\left(1 - 2\frac{r}{u}\left[\xi - \frac{t}{2\lambda\zeta}\right] + \frac{r^2}{u^2}\right)^{\lambda}} , \tag{5.10}$$

and writing $\mathcal{G}[F]$ as,

$$\mathcal{G}[F] = -\frac{1}{4\pi^2} \int_{|\zeta| = \epsilon_1} \frac{d\zeta}{\zeta} \int_{|u| = \epsilon_2} K_3\left(\frac{r}{u}, \xi, \frac{t}{\zeta}\right) F(u, \zeta) \frac{du}{u} . \tag{5.11}$$

The solutions $V(r, \xi, t)$ are defined in the small of the origin; however, they may be continued by deforming the contours of integration in the usual manner [5][15].

We make a few concluding remarks about the process of inverting the double summations over m and n, which were used in this and the preceeding sections to find a closed form for the generating kernels. We recall, first, that the generating function series expansion for the $\{r^n C_n^\lambda(\xi)\}$ are valid for all complex r, such that $|r| < \min|\xi \pm (\xi^2-1)^{1/2}|$. (For x, y real, $0 \leq \xi \leq 1$, and $|r| < 1$) The double series in (5.3) may be considered for each fixed ξ as representing an analytic function of the two complex variables r, t which converges uniformly and absolutely in a suitably small bi-cylindrical neighborhood of the origin in complex r, t space. Since we have absolute convergence in this bi-cylinder for a double power series, we may interchange orders of summation there, and either summation yields the same analytic function of two complex variables [19].

REFERENCES

1. Bergman, S., Zur Theorie der ein-und mehrwertigen harmonischen Funktionen des drei dimensional Raumes, Math. Zeit. 24 (1926) pp. 641-669.

2. Bergman, S., Zur Theorie des algebraischen Potential Funktionen des drei dimensional Raumes, Math. Ann. 99 (1928) pp. 629-659, and 101 (1929) pp. 534-538.

3. Bergman S., Residue theorems of harmonic functions of three variables, Bull. Amer. Math. Soc. 49, 2, (1943) pp. 163-174.

4. Bergman S., Integral Operators in the Theory of Linear Partial Differential Equations, Ersebn. Math. U. Grenzgeb. 23 Springer, Berlin (1960).

5. Gilbert, R. P., Singularities of three-dimensional harmonic functions, Pac. J. Math. 10 (1961) pp. 1243-1255.

6. Whittaker, E. T., and Watson, G. N., A Course of Modern Analysis, Cambridge (1952).

7. Erdeyli, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., Higher Transcendental Functions, II, McGraw Hill, New York, (1953).

8. Erdeyli, A. et. all., Higher Transcendental Functions, I, McGraw Hill, New York, (1953).

9. Hobson, E. W., The Theory of Spherical and Ellipsoidal Harmonics, Cambridge (1931).

10. Kreyszig, E., On regular and singular harmonic functions of three variables, Arch. Rat. Mech. Anal. 4 (1960) pp. 352-370.

11. Kreyszig, E., Kanonische Integraloperatoren zur erzeugung harmonischer Funktionen von vier Veraenderlichen, to appear Arch Mathematik.

12. Gilbert, R. P., A note on the singularities of harmonic functions in three variables, Proc. Amer. Math. Soc. 13, 2, (1962) pp. 229-232.

13. White, A., Singularities of harmonic functions of three variables generated by Whittaker-Bergman operators, Annales Polonici Mathematici, 10 (1961) pp. 81-100.

14. White, A., Singularities of a harmonic function of three variables given by its series development, to appear Pac. Journal Math. also MRC Technical Summary Report #312, April 1962.

15. Gilbert, R. P., Singularities of solutions to the wave-equation in three dimensions, J. Reine u. Angew. Math. 205 (1960) pp. 75-81.

16. Gröbner, W., and Hofreiter, N., Itegraltafel, erster teil, Springer Verlag, Wien und Innsbruch, (1949).

17. Weinstein, A., Generalized axially symmetric potential theory, Bulletin American Math. Soc., 59 (1953) pp. 20-38.

18. Weinstein, A., Discontinuous integrals and generalized potential theory, Trans. Amer. Math. Soc., 63 (1948) pp. 342-354.

19. Osgood, W. F., Lehrbuch der Funktionen Theorie, zweiter Band, erste Lieferung, Mathematischen Wissenschaften, XX, Teubner, Berlin, (1924).